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The Cartesian Oval and the Elliptic Functions p and σ.

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1911.

A dissertation submitted to the Board of University Studies of the Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

On Cartesian Ovals and the Elliptic Functions $\wp$ and σ.

The object of this paper is to show that the leading properties of the Cartesian Ovals are developed in such text as Williamson's Differential Calculus, Salmon's Higher Plane Curves or Loria's Spezielle Algebraische und Transcendente Ebene Kurven may be readily deduced from the Weierstrass elliptic function $\wp u$ and that they give a geometric interpretation to these formulæ.

Greenhill has already pointed this out in a paper (Proc. London Math. Soc., t. XVII, 1886) and also in his treatise on Elliptic Functions, but

from the standpoint of the Jacobi functions which do not so readily lend themselves to interpretation.

Professor Frank Morley published a note on the subject some years ago in the Harvard Studies, and the present discussion is based upon lectures given by Professor Morley at Johns Hopkins University. [See also his Treatise on the Theory of Functions, P.S.K.]

When the u-plane is mapped on the x-plane by means of the equation

$$du = \frac{dx}{\sqrt{Q}},$$

where Q is a quartic in x, to lines in the u-plane parallel to the real or imaginary axis, correspond in the x-plane bicircular quartics whose real foci are the zeros of Q, provided these zeros are concyclic

or anti-cyclic. [Greenhill, Camb. Phil. Proc., t. IV.; Franklin, American Journal, XI. 3 and XII. 4.]

The problem of reducing this curve to the Weierstrass form, which we use when one focus is at infinity, is analogous to the reduction of the bicircular quartic to the Cartesian. [Salmon, Higher Plane Curves, §280.]

1. Preliminary examples of mapping.

We let

$$x = pu.$$

Then

$$\frac{dx}{du} = p'u = -\sqrt{4p^3u - g_2 pu - g_3} = -\sqrt{pu-e_1 \cdot pu-e_2 \cdot pu-e_3}$$

[Hawkins and Morley, Introduction to Analytic Functions, p. 259, (16).]

Hence $$du = -\frac{dx}{\sqrt{x-e_1 \cdot x-e_2 \cdot x-e_3}}.$$

We suppose g_2 and g_3 to be real and $\Delta = g_2^3 - 27 g_3^2 > 0$.

Then e_1, e_2 and e_3 are real.

Let $e_1 > e_2 > e_3$.

Since when $x < e_3$ du/dx is imaginary and when $x > e_1$ du/dx is real, we have a real period $2\omega_1$ and a purely imaginary period $2\omega_3$. Hence the parallelogram of periods is a rectangle.

We write for symmetry

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

Let u describe some path in the u-plane and trace the corresponding path of x on the x-plane.

For example let u move from 0 to ω_1, along the real axis.

Then for x is real and decreases

from infinity to e_1.

If w bends to the left to avoid the critical point e_1 in a quarter circle about it, x turns to the left and describes a semi-circle about e_1, as the isogonality breaks down at this point and the angle which w makes about e_1 is one-half that which x makes about e_1.

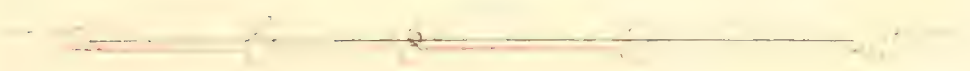
As w moves parallel to the axis of imaginaries from w_1 to $-w_2$, x goes from e_1 to e_2. As w turns again to the left making a quarter circle about $-w_2$, x makes a semi-circle about e_2 and goes from e_2 to e_3 while w moves parallel to the real axes from $-w_2$ to w_3. As w passes about w_3 into quadrant to the left, x passes around e_3 in a semi-circle and

goes from ℓ_3 to minus infinity as u passes from ω_1 to 0.

That is, as u decreases the rectangle σ_5 the parallelogram $0, \omega_1, \omega_2, \omega_3$ traverses the real axis from β to infinity to minus infinity. [Burkhardt, Functionentheorie, Vol. 3. §86. IX.]

u-plane.

x plane.



As u goes from 0 to $2w$, passing to the left in a semi-circle about c_1, v moves from plus infinity to c, then in a circle to the left about c, and back to plus infinity.

Any point within the first named parallelogram $0, w_1, -w_2, w_3$ is mapped into a point on the negative half of the v-plane. The lines of the mesh

period parallelogram doubly covers the x-plane, since $\wp u$ is an elliptic function of the second order.

2. The map of a critical line and the equation of the curve.

Suppose u to describe a line parallel to the axis of imaginaries and at a distance α from it.

Let

$$u = \alpha + i\beta \text{ and}$$
$$\bar{u} = \alpha - i\beta.$$

Then the equation of the line is

(1) $$u + \bar{u} = 2\alpha,$$

u and $\bar{u}$ being conjugate and α real.

Since

$$\wp u = \frac{1}{u^2} + \frac{g_2 u^2}{2^2 \cdot 5} + \frac{g_3 u^6}{2^2 \cdot 7} + \cdots$$ [Tannery and Molk, Theorie des Fonctions Elliptiques, IX. 3.]

and g_2 and g_3 are real by hypothesis,

(2) $\quad \begin{cases} X = \wp(\alpha + i\beta) = \wp u \text{ and} \\ \bar{X} = \wp(\alpha - i\beta) = \wp \bar{u} \end{cases}$

are conjugate complex numbers.

[Burnside, Ell. Fns. I §71]

To find the curve in x which corresponds to the vertical through x trace by, eliminate β between equations (1) and (2) by means of an addition theorem which gives the relation between $\wp(u+v)$, $\wp u$ and $\wp v$.

To obtain the theorem we have

$$\wp'^2 u = 4\wp^3 u - g_2 \wp u - g_3.$$

Consider

$$\wp' u - C \wp u - C',$$

where C and C' are constants.

It is an elliptic function with the same periods as $\wp u$.

At $u=0$ are infinity of the third order at $u=0$.

Therefore it has three zeros u, v and w such that

$$u+v+w=0.$$

For each of these points

$$(C\wp u+C')^2=(\wp'u)^2=4\wp^3u-g_2\wp u-g_3, \text{ or}$$
$$4\wp^3u-C^2\wp^2u-(2CC'+g_2)\wp u-(C'^2+g_3)=0.$$

Then

$$S_1\equiv \wp u+\wp v+\wp w=\tfrac{1}{4}C^2,$$
$$S_2\equiv \wp u\cdot\wp v+\wp u\cdot\wp w+\wp v\wp w=-\tfrac{1}{4}(2CC'+g_2),$$
$$S_3\equiv \wp u\cdot\wp v\cdot\wp w=\tfrac{1}{4}(C'^2+g_3).$$

Therefore

$$S_1(S_3-\tfrac{1}{4}g_3)=\tfrac{1}{16}C^2C'^2=\tfrac{1}{4}(S_2+\tfrac{1}{4}g_2)^2.$$

From which we get an addition Theorem:

$$(S_2+\tfrac{1}{4}g_2)^2=S_1(4S_3-g_3).$$

In this theorem substitute

$u = \alpha + i\beta$

$v = \alpha - i\beta$, whence

$w = -2\alpha$.

If we take α merely, write A for $p\,2\alpha$ or $p(-2\alpha)$. Then

$S_1 = x + \bar{x} + A$

$S_2 = x\bar{x} + A(x + \bar{x})$

$S_3 = Ax\bar{x}$

Therefore

(3) $\left[x\bar{x} + A(x+\bar{x}) + \frac{1}{4}g_2\right]^2 = (x+\bar{x}+A)(4Ax\bar{x} - g_2)$.

The coefficient of x^2 is

$(\bar{x}+A)^2 - 4A\bar{x} \equiv (\bar{x}-A)^2$.

Speaking roughly, this gives the tangents at the circular points δ or Δ and the equation is that of a bicircular quartic, with cusps at δ and Δ, that is, a Cartesian Oval.

The cusp tangents at $\mathcal{L}$ and $\mathcal{J}$,

$$x - A = 0 \quad \text{and}$$
$$\bar{x} - A = 0,$$

meet at the triple focus A corresponding to the point

$$u = 2\alpha$$

To obtain the angle foci, we have the two equations:

x-discriminant $= 0$.

$\bar{x}$-discriminant $= 0$.

Rearranging the equation of the curve according to the powers of $\bar{x}$, we have

$$\bar{x}^2(x-A)^2 + \bar{x}\left(\tfrac{1}{2}g_2 x + \tfrac{1}{2}g_2 A - 2Ax^2 - 2A^2 x + g_3\right) + A^2x^2 + \tfrac{1}{2}g_2 Ax + \tfrac{1}{16}g_2^2 + g_3 x + g_3 A = 0.$$

Therefore, the $\bar{x}$-discriminant is

$$\left(\tfrac{1}{2} g_2 x + \tfrac{1}{2} g_2 A - 2Ax^2 - 2A^2 x + g_3\right)^2 =$$
$$4(x-A)^2\left(A^2x^2 + \tfrac{1}{2} g_2 A + \tfrac{1}{16} g_2^2 + g_3 x + g_3 A\right). \quad \text{or}$$

$$^{*}\left[4A^3 - g_2 A - g_3\right]\left[4x^3 - g_2 x - g_3\right] = 0.$$

Hence the single foci are e_1, e_2 and e_3.

The curve depends on the single constant $\wp 2\alpha$ or A. It will therefore include the representation of four lines in the rectangle; namely, those which meet the real axis at the four points α, $\omega' - \alpha$, $\omega, + \alpha$, $2\omega' - \alpha$.

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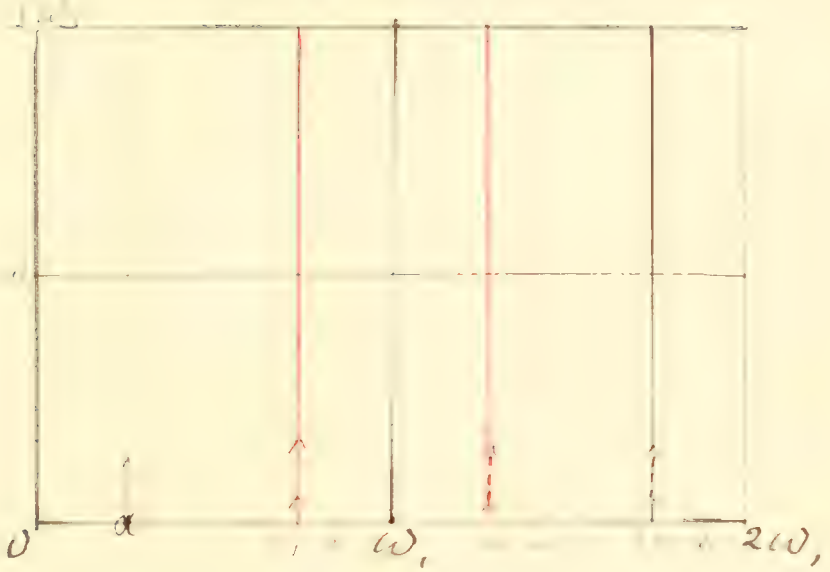
* If $\alpha = \frac{1}{2}\omega$, and so $\wp 2\alpha$ or A comes the first factor of this equation to vanish, the Cartesian oval becomes a circle counted twice (see page 17) and the foci accordingly are arbitrary.

u – plane.



x – plane.

If u is any one of these points,

$$\wp 2u = \wp 2\alpha.$$

Let $i\beta = v$

Since $\wp(\alpha+v) = \wp[2\omega_1 - (\alpha+v)] = \wp(2\omega_1 - \alpha - v)$,

the outer oval corresponds to the first and fourth line, the $\wp$-function moving in clockwise direction from the real axis as v moves up from α and in anti-clockwise direction as it moves up from $2\omega_1 - \alpha$.

Similarly the inner oval taken twice, in opposite direction, corresponds to the second and third line.

In the same way, to lines parallel to the real axis correspond Cartesian ovals with the same foci as may be seen by interchanging

x and v in this discussion.
Since at every point $u = (x + iv)$ in the u-plane there is a line parallel to the axis of u's and one parallel to the axis of imaginaries, 2 through every point in the p-plane pass two orthogonal curves or at corresponding to these two lines.

3. The invariants of the curve and its $\wp$-function

Among the formulae for the $\wp$-function is

$$[\wp(u+\omega_1-e_1][\wp u - e_1] = e_2 - e_1 \cdot e_3 - e_1$$

[Hankins and Morley, Introduction. p 261, (20).]

Therefore, if by quasi-inversion we mean ordinary inversion followed by reflexion in the real axis, $\alpha+v+\ldots$, is the quasi-inverse of $\alpha+v$, and the inverse of $\alpha-v$ since the curve is symmetric with respect to the real axis. Hence the curve is its own inverse with respect to a circle with center e_1 and radius $\sqrt{e_2-e_1}$, $\sqrt{e_3-e_1}$, (a circle with radius real).

As α approaches $\pm w_1$, the two verticals through α and $w_1-\alpha$ approach coincidence. Therefore, if

$$\alpha = \tfrac{1}{2} w_1,$$

the two ovals coincide and each is the above mentioned circle.

[to commence the discussion of curves on the curve by its parameter.]

An algebraic proof of the last statement on page 17 may be added.

Substituting in equation (3) p. 11

$2\alpha = \omega_1$ and therefor

$\wp 2\alpha = e_1$ and remembering that

$-\frac{1}{4} g_2 = \Sigma e_1 e_2$ and

$g_3 = 4 e_1 e_2 e_3$, we get

$$[x\bar{x} + (x+\bar{x}) e_1 - \Sigma e_1 e_2]^2 = (x+\bar{x}+e_1)(4 e_1 x\bar{x} - 4 e_2 e_3),$$

or

$$[x\bar{x} + (x+\bar{x}+e_1) e_1 - e_2 e_3]^2 = 4 e_1 (x+\bar{x}+e_1)(x\bar{x} - e_2 e_3).$$

Since

$$[x\bar{x} - e_2 e_3 - e_1 (x+\bar{x}+e_1)]^2 =$$

$$[x\bar{x} - e_1 (x+\bar{x}) - e_1^2 - e_2 e_3]^2 =$$

$$[(x-e_1)(\bar{x}-e_1) - (2e_1^2 + e_2 e_3)]^2 =$$

$$[(x-e_1)(\bar{x}-e_1) - (e_2-e_1)(e_3-e_1) = 0,$$

which is a circle with center e_1 and radius $\sqrt{e_2-e_1} \cdot \sqrt{e_3-e_1}$.

Likewise

$$[p(u+\omega_2)-e_2][pu-e_2]=e_1-e_2\cdot e_3-e_2,$$

or more generally

$$[p(u+\omega_\lambda)-e_\lambda]=e_\mu-e_\lambda\cdot e_\nu-e_\lambda$$

$(\lambda,\mu,\nu=1,2,3.)$

Thus the curve has three inversion circles, one around each focus as center. (But the circle with e_2 as center is multipartite, since its radius

$\sqrt{e_3-e_2}\cdot\sqrt{e_1-e_2}$

is imaginary.)

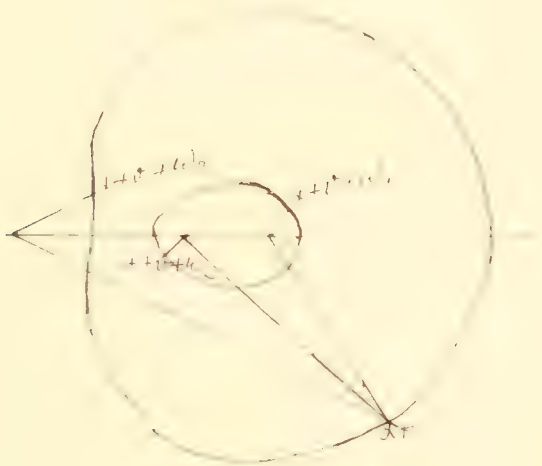
These three circles are orthogonal since

$$e_\mu-e_\lambda\cdot e_\nu-e_\lambda+e_\lambda-e_\nu\cdot e_\mu-e_\nu=(e_\nu-e_\lambda)^2.$$

But the bicircular quartic in general inverts into itself from any one of its four foci, and

the four circles of inversion are all orthogonal. The fourth focus of the Cartesian oval is at infinity and the real axis is the circle of infinite radius orthogonal to the other three circles of inversion. Hence inversion with respect to the fourth focus amounts to reflexion in the real axis.

[S. Roberts, Proc. Math. Soc. London, t III, p 108.]



At first I thought it might appear possible to obtain any number of points on the curve from a given point by means of repeated quasi-inversions. But in fact we get a group of only four points. For, if

$$\left[\wp(u+w_\lambda)-e_\lambda\right]\left[\wp u-e_\lambda\right]=\left[e_u-e_\lambda\cdot e_\lambda-e_\lambda\right] \text{ and}$$

$$\left[\wp(u+w_\mu)-e_\mu\right]\left[\wp u-e_\mu\right]=\left[e_\nu-e_\mu\cdot e_\lambda-e_\mu\right] \text{ and}$$

$$\left[\wp(u+w_\nu)-e_\nu\right]\left[\wp u-e_\nu\right]=\left[e_\mu-e_\nu\cdot e_\lambda-e_\nu\right],$$

we get, by eliminating $\wp u$ from the second and third equations,

$$\left[\wp(u+w_\mu)-e_\lambda\right]\left[\wp(u+w_\nu)-e_\lambda\right]=\left[e_u-e_\lambda\cdot e_\nu-e_\lambda\right]$$

Hence if $p u$ and $p(u+\omega_\lambda)$ are quasi-inverse as to ℓ_λ, then $p(u+\omega_\mu)$ and $p(u+\omega_\nu)$ are also quasi-inverse as to ℓ_λ.

If we call inversion as to ℓ_λ α_λ $(\lambda = 1\,2\,3)$ and reflexion in the real axis α_4 and let

$$A = \alpha_1\alpha_4 = \alpha_4\alpha_1 = \alpha_2\alpha_3 = \alpha_3\alpha_2,$$

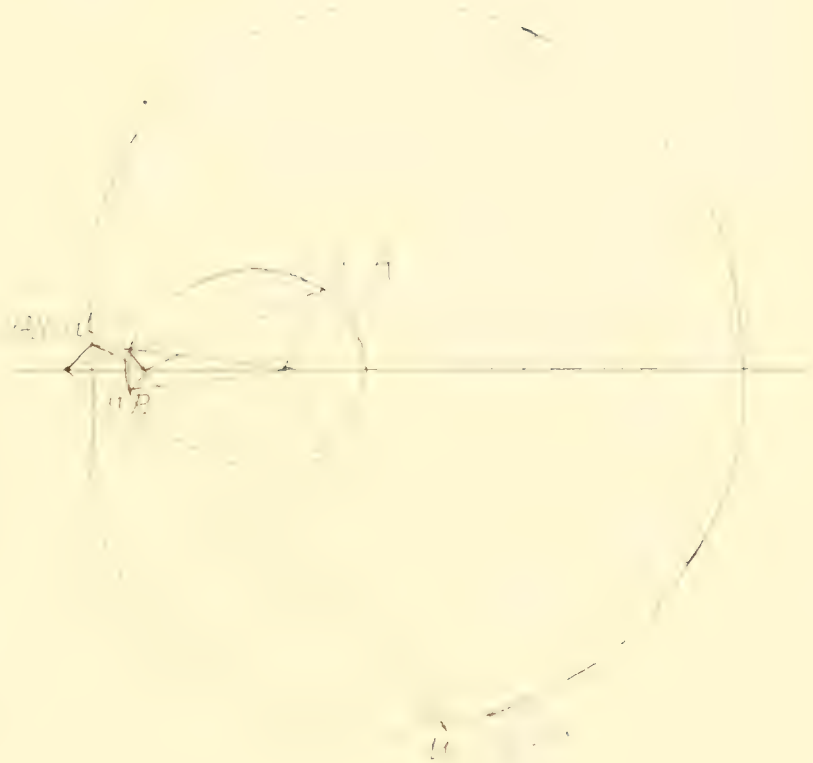
$$B = \alpha_2\alpha_4 = \alpha_4\alpha_2 = \alpha_1\alpha_3 = \alpha_3\alpha_1,$$

$$C = \alpha_3\alpha_4 = \alpha_4\alpha_3 = \alpha_1\alpha_2 = \alpha_2\alpha_1,$$

where A B and C are evidently the three quasi-inversions there "$1, A, B, C \equiv AB$ form a four group"

$$1 = A^2 = B^2 = (AB)^2.$$

[Klein, Ikosaeder, §5 and §13. Cayley, Math. Papers vol. II p. 125



If we reflect these four points in the real axis we get four other points which, together with these, are the eight points of intersection of this curve with the confocal orthogonal Cartesian ovals which pass through the point $\alpha+i\beta$. That is the curve which is the map of the line in the w-plane through $\alpha+i\beta$ parallel to the real axis.

4. Focal Radii.

If ρ_1 and ρ_2 are the focal radii of the curve,

$$k\rho_1 + \rho_2 = \text{constant}$$

is a well known property of the curve, which may be derived

in the following manner.

A focal vector of any point on the curve is

$$r_1 = p(\alpha+v) - e_1,$$

and of the conjugate point is

$$\bar{r}_1 = p(\alpha-v) - e_1.$$

Then the focal radius is

$$\rho_1 = \sqrt{r_1 \bar{r}_1} = \sqrt{p(\alpha+v)-e_1}\,\sqrt{p(\alpha-v)-e_1}.$$

But $\sqrt{pu-e_1} = \frac{\sigma_1 u}{\sigma u}$.

[Hurwitz and Courant, Theory of Functions, p. 413]

Hence

$$\rho_1 = \frac{\sigma_1(\alpha+v)\cdot\sigma_1(\alpha-v)}{\sigma(\alpha+v)\,\sigma(\alpha-v)}.$$

Thus ρ_1 is an elliptic function with the same periods $2\omega_1, 2\omega_2$ and $2\omega_3$ of the second order with unit

at $\pm x$ and residue $\mp \frac{\sigma_1 2x}{\sigma 2\alpha}$.

Likewise

$$\rho_2 = \frac{\sigma_2(\alpha+v)\,\sigma_2(\alpha-v)}{\sigma(\alpha+v)\,\sigma(\alpha-v)}$$

is an elliptic function with infinities at $\pm x$ and residue $+\frac{-\sigma_2 2x}{\sigma 2\alpha}$

Let the distance to the triple focus pass from a single focus e_1 be λ_1.

Then

$$\rho 2x - e_1 = \lambda_1,$$
$$\sqrt{\rho 2x - e_1} = \sqrt{\lambda_1} = \frac{\sigma_1 2x}{\sigma 2\alpha}.$$

Hence the residues

of ρ_1 at $\pm x$ are $\mp 1, -1$,
of ρ_2 at $\pm x$ are $\mp \sqrt{\lambda_2}$.

Now if two homogeneous elliptic functions ρ and ρ' of the second order have the same infinities c and c' but different residues γ_1 and γ_2

at an infinite c, then

$$\begin{vmatrix} f,u & r_1 \\ f\alpha u & r_2 \end{vmatrix} = \text{Constant}.$$

[Hardness and Mordy, Introduction to Analytic Function, P 259 II.]

Thus :-

$$\begin{vmatrix} p_1 & -\sqrt{\lambda_1} \\ p_2 & -\sqrt{\lambda_2} \end{vmatrix} = k.$$

that is there is a linear relation between the focal radii.

1- proceed to find k.

$$\sqrt{\lambda_1}\sqrt{p(\alpha+v)-e_2}\sqrt{p(\alpha-v)-e_2} - \sqrt{\lambda_2}\sqrt{p(\alpha+v)-e_1}\sqrt{p(\alpha-v)-e_1}$$

2- v approaches α then

$$\sqrt{\lambda_1}\sqrt{p\alpha - e_2 + p'\alpha(v-\alpha)\cdots}\sqrt{\frac{1}{(\alpha-v)^2}\cdots - e_2}$$

$$-\sqrt{\lambda_2}\sqrt{p\alpha - e_1 + p'\alpha(v-\alpha)\cdots}\sqrt{\frac{1}{(\alpha-v)^2}\cdots - e_1} = k$$

$$\frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{\alpha - v}\left[1 + \frac{1}{2}\frac{p'\alpha}{\lambda_2}(v-\alpha) + P(v-\alpha)\right]$$

$$-\frac{\sqrt{\lambda_2}\sqrt{\lambda_1}}{\alpha - v}\left[1 + \frac{1}{2}\frac{p'\alpha}{\lambda_1}(v-\alpha) + P(v-\alpha)\right] = k$$

Theorem as it approaches a

$$\frac{1}{2}\sqrt{\lambda_1}\sqrt{\lambda_2}\left[\frac{p'^2\alpha}{\lambda_1} - \frac{p'^2\alpha}{\lambda_2}\right] = \kappa$$

or since $p'^2\alpha = -2\sqrt{\lambda_1\lambda_2\lambda_3}$

$$\kappa = \lambda_1\sqrt{\lambda_3} - \lambda_2\sqrt{\lambda_3}.$$

Hence

$$\sqrt{\lambda_1}\,\rho_2 - \sqrt{\lambda_2}\,\rho_1 = \sqrt{\lambda_3}\,(\lambda_1 - \lambda_2)$$

or more generally

$$\sqrt{\lambda_\nu}\,\rho_\mu - \sqrt{\lambda_\mu}\,\rho_\nu = \sqrt{\lambda_\lambda}\,(\lambda_\nu - \lambda_\mu).$$

Two of the three such equations give

$$\sqrt{\lambda_1}\sqrt{\lambda_2}\rho_3 - \lambda_1\lambda_2 = \sqrt{\lambda_2}\sqrt{\lambda_3}\rho_1 - \lambda_2\lambda_3 = \sqrt{\lambda_3}\sqrt{\lambda_1}\rho_2 - \lambda_3\lambda_1.$$

From these equations one obtains Poncelet's form [Nouv. Ann. 2ème Sér. 2622, Ed. Terquem t. XI, p. 56.]

$$\sqrt{\lambda_1}(\lambda_2 - \lambda_3)\rho_1 + \sqrt{\lambda_2}(\lambda_3 - \lambda_1)\rho_2 + \sqrt{\lambda_3}(\lambda_1 - \lambda_2)\rho_3 = 0.$$

The bifocal equation of the Cartesian oval is commonly given, as

$$l\rho_1 \pm m\rho_2 = k,$$

[Williamson, Diff. Cal., p. 375.]

in which the upper sign belongs to the equation of the inner oval and the lower to that of the outer oval.

Since for the outer oval, which may $\alpha + \iota$, ι varying from 0 to 2ω, each of the quantities

$$\rho_1, \rho_2, \sqrt{\lambda_1}, \text{ and } \sqrt{\lambda_2}$$

is positive, our equation

$$\sqrt{\lambda_2}\rho_1 - \sqrt{\lambda_1}\rho_2 = \sqrt{\lambda_2}\lambda_2 - \lambda_1,$$

is consistent with that given above.

That it holds also for the inner oval may be explained in either of two ways.

a' We may consider the inner oval as the $\wp$ map of $\alpha' + v$ where

$$\alpha' = \alpha + \omega_1 .$$

In that case

$$\rho_1 , \rho_2 \text{ and } \sqrt{\lambda_2} = \frac{\sigma_2 2\alpha'}{\sigma 2\alpha'}$$

remain positive but

$\sqrt{\lambda_1} = \frac{\sigma_1 2\alpha'}{\sigma 2\alpha'}$ has passed a zero (ω_1) of $\sigma_1 2\alpha$ and is therefore negative. Hence the equation agrees in sign with the one usually given.

b We may consider α as fixed and take as equations of the inner oval

$$x = \wp(\alpha + \omega_1 + v),$$
$$\bar{x} = \wp(\alpha - \omega_1 - v),$$

in which case $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ are

unchanged and must be taken positively.

To study the signs of ρ_1 and ρ_2 let us look at the limiting case when V approaches zero.

Then

$$\rho_1 = \frac{\sigma_1(\alpha+\omega_1)\,\sigma_1(\alpha-\omega_1)}{\sigma(\alpha+\omega_1)\,\sigma(\alpha-\omega_1)}$$

is positive since $\sigma_1(\alpha+\omega_1)$ is negative, $\sigma_1(\alpha-\omega_1)$ positive, $\sigma(\alpha+\omega_1)$ positive and $\sigma(\alpha-\omega_1)$ negative. [Tannery and Molk, Theorie des Fonctions Elliptiques, p. 201.]

But

$$\rho_2 = \frac{\sigma_2(\alpha+\omega_1)\,\sigma_2(\alpha-\omega_1)}{\sigma(\alpha+\omega_1)\,\sigma(\alpha-\omega_1)}$$

is negative as $\sigma_2(\alpha+\omega_1)$ and $\sigma_2(\alpha-\omega_1)$ are both positive.

Therefore our form of the equation holds equally well for the inner oval.

Similarly it may be shown that each of the expression

$$\sqrt{\lambda_x}\sqrt{\lambda_y}\rho_r - \lambda_x\lambda_u = \tfrac{1}{2}[\rho^2 - (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda\lambda_1)],$$

where ρ is the distance of any point on the curve from the triple focus. For

$$\rho^2 = [p(\alpha+v) - p2\alpha][p(\alpha-v) - p2\alpha]$$

which is an elliptic function of the second order with infinities at $\pm\alpha$ and residues $\pm p'2\alpha$.

Hence

$$\begin{vmatrix} \rho_1 & -\sqrt{\lambda_1} \\ \rho^2 & p'2\alpha \end{vmatrix} = \kappa,$$

or

$$\rho_1\, p'2\alpha + \rho^2\sqrt{\lambda_1} = \kappa,$$

or

$$\kappa = p'2\alpha\sqrt{p(\alpha+v)-e_1}\sqrt{p(\alpha-v)-e_1} + \sqrt{\lambda_1}[p(\alpha+v)-p2\alpha][p(\alpha-v)-p2\alpha].$$

To find κ let v approach α

$$p'^2\alpha\sqrt{p^2\alpha+(v-\alpha)p'^2\alpha\cdots-e_1}\sqrt{\frac{1}{(\alpha-v)^2}-\cdots-e_1}$$
$$+\sqrt{\lambda_1}\left[p^2\alpha+(v-\alpha)p'^2\alpha+\frac{(v-\alpha)^2}{2}p''^2\alpha\cdots-p^2\alpha\right]\left[\frac{1}{(\alpha-v)^2}\right]=\kappa$$

or

$$p'^2\alpha\frac{\sqrt{\lambda_1}}{\alpha-v}\left[1+\frac{1}{2}\frac{p^2\alpha(v-\alpha)}{\lambda_1}\right]+\frac{\sqrt{\lambda_1}\,p'^2\alpha}{v-\alpha}+\frac{\sqrt{\lambda_1}\,p'^2\alpha}{2}=\kappa$$

Hence

$$\kappa=-\frac{\sqrt{\lambda_1}}{2}\frac{p'^2\alpha}{\lambda_1}+\sqrt{\lambda_1}\,\frac{p''^2\alpha}{2}.$$

But since

$$p'^2\alpha=4(p^2\alpha-e_1)(p^2\alpha-e_2)(p^2\alpha-e_3)$$
$$2\,p'^2\alpha\cdot p''^2\alpha=4p'^2\alpha\left[(p^2\alpha-e_2)(p^2\alpha-e_3)\right.$$
$$\left.+(p^2\alpha-e_3)(p^2\alpha-e_1)+(p^2\alpha-e_1)(p^2\alpha-e_2)\right.$$

Therefore

$$p''^2\alpha=2(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)$$

and

$$\kappa=-2\sqrt{\lambda_1}\,\lambda_2\lambda_3+\sqrt{\lambda_1}(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1).$$

Hence

$$\rho_1\sqrt{\lambda_2\lambda_3}-\lambda_2\lambda_3=\frac{1}{2}\left[\rho^2-(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\right].$$

Similarly we may obtain a relation between the distances of any point on the curve from the double tangent and from a single focus.

The equation of the double tangent is

$$X + \bar{X} + A = 0,$$

as may be seen from equation (3 page 11, [where $\alpha + \beta = p_2 x = p_1 x$])

If we denote by T the distance to a point of the curve from the double tangent then

$$2T = p(\alpha + v) + p(\alpha - v + A.$$

But if

$$u + v + w = 0,$$

$$pu + pv + pw = \frac{1}{4}\left(\frac{p'u - p'v}{pu - pv}\right)^2 = (\xi u + \xi v + \xi w)^2.$$

[Hawkins and Morley, Treatise on Theory of Func. p 303 & p 304]

Therefore

$$\sqrt{2\sigma} = \zeta(\alpha+v)+\zeta(\alpha-v)-\zeta 2\alpha = -\frac{1}{2}\left[\frac{p'(\alpha+v)-p'(\alpha-v)}{p(\alpha+v)-p(\alpha-v)}\right]$$

which is an elliptic function of the second order with infinities at $\pm v$ and $\pm\alpha$ and residues ∓ 1.

Hence

$$\begin{vmatrix} \sqrt{2\sigma} & 1 \\ p_1 & \sqrt{\lambda_1} \end{vmatrix} = \kappa,$$

or

$$\sqrt{\lambda_1}\left[\zeta(\alpha+v)+\zeta(\alpha-v)-\zeta 2\alpha\right]-\sqrt{p(\alpha+v)-e_1}\sqrt{p(\alpha-v)-e_1} = \kappa.$$

To find κ let v approach α then

$$\sqrt{\lambda}\left[\zeta 2\alpha - p 2\alpha (v-\alpha)\ldots - \frac{1}{v-\alpha} \cdots - \zeta 2\alpha\right]$$
$$-\sqrt{p 2\alpha + p' 2\alpha (v-\alpha \cdots - e_1}\sqrt{\frac{1}{(\alpha-v)^2} - \cdots - e_1} = \kappa,$$

or

$$\frac{\sqrt{\lambda_1}}{\alpha-v} \cdots - \frac{\sqrt{\lambda_1}}{\alpha-v}\left[1 + \frac{1}{2}\frac{p 2\alpha}{\lambda_1}(v-\alpha)\cdots\right] = \kappa.$$

Therefore

$$\kappa = \frac{1}{2}\frac{p' 2\alpha}{\sqrt{\lambda_1}} = -\frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{\lambda_1}} = -\sqrt{\lambda_2\lambda_3}\,.$$

Hence

$$\sqrt{\lambda_1}\sqrt{2T} = \rho_1 - \sqrt{\lambda_2\lambda_3},$$

or

$$\sqrt{\lambda_1}\sqrt{2T} = \rho_2 - \sqrt{\lambda_4\lambda_5}.$$

From these we could have obtained the formulae derived on page 29,

$$\sqrt{\lambda_1}\sqrt{\lambda_2}\rho_3 - \lambda_1\lambda_2 = \sqrt{\lambda_2}\sqrt{\lambda_3}\rho_1 - \lambda_2\lambda_3 = \sqrt{\lambda_3}\sqrt{\lambda_1}\rho_2 - \lambda_3\lambda_1.$$

5. <u>Line-the definition of the curve.</u>

The equation of the tangent to the curve at any point $p(\alpha+v)$ is

$$\begin{vmatrix} X & \bar{X} & 1 \\ p(\alpha+v) & p(\alpha-v) & 1 \\ p'(\alpha+v) & -p'(\alpha-v) & 0 \end{vmatrix} = .$$

Hence the equation of the normal at the same point is

$$\begin{vmatrix} x & \bar{x} & 1 \\ p(\alpha+v & p(\alpha-v) & 1 \\ p'(\alpha+v & p'(\alpha-v) & 0 \end{vmatrix} = 0.$$

To find the point N where the normal cuts the real axis, let

$$x = \bar{x} = N.$$

Hence

$$N = p(\alpha+v) - p'(\alpha+v\left[\frac{p(\alpha+v) - p(\alpha-v}{p'(\alpha+v| - p'(\alpha-v|}\right].$$

But, since

$$\begin{vmatrix} 1 & p(\alpha+v) & p'(\alpha+v \\ 1 & p(\alpha-v) & p'(\alpha-v \\ 1 & p\,2\alpha & -p'2\alpha \end{vmatrix} = 0.$$

[Burkhardt, Functionen Theorie, II, p. 62.]

or

$$p(\alpha+v) = p2\alpha + \left[p'2\alpha + p'(\alpha+v)\right]\left[\frac{p(\alpha+v) - p(\alpha-v)}{p'(\alpha+v) - p'(\alpha-v)}\right]$$

$$N = p2\alpha + p'2\alpha\left[\frac{p(\alpha+v) - p(\alpha-v)}{p'(\alpha+v) - p'(\alpha-v)}\right].$$

But from page 36,

$$-\frac{1}{2}\left[\frac{p'(\alpha+v) - p'(\alpha-v)}{p(\alpha+v) - p(\alpha-v)}\right] = \sqrt{2J}.$$

Hence

$$N = p2\alpha - \frac{p'2\alpha}{2\sqrt{2J}} = p2\alpha + \frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2J}},$$

or

$$N - e_1 = \lambda_1 + \frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2J}} = \frac{\sqrt{\lambda_1}\left(\sqrt{2\lambda_1 J} + \sqrt{\lambda_2\lambda_3}\right)}{\sqrt{2J}}.$$

Therefore

$$N - e_1 = \frac{\sqrt{\lambda_1}}{\sqrt{2J}}\,P_1, \qquad [\text{see page } 37]$$

and

$$N - p2\alpha = \frac{\sqrt{\lambda_1\lambda_2\lambda_3}}{\sqrt{2J}}.$$

Therefore

$$\frac{N - e_1}{N - p2\alpha} = \frac{\sqrt{\lambda_2\lambda_3}}{P_1}.$$

If we draw from p_{2x} a line perpendicular to the tangent at x (and therefore parallel to the normal) cutting ρ_1 at $\bar{\tau}_1$, then

$$\frac{x\bar{\tau}_1}{\rho_1} = \frac{x - p_{2x}}{x - c_1} = \frac{\sqrt{\lambda_2\lambda_3}}{\rho_1}.$$

Therefore

$$x\bar{\tau}_1 = \sqrt{\lambda_2\lambda_3}.$$

Similarly

$$x\bar{\tau}_2 = \sqrt{\lambda_3\lambda_1}$$

and

$$x\bar{\tau}_3 = \sqrt{\lambda_1\lambda_2}.$$

Hence we derive the theorem:—

The perpendicular from the triple focus on the tangent at a point x of a Cartesian oval meets a focal radius ρ_x at a point τ_x such that

$$|x - \tau_x| = \text{a constant}.$$

Conversely if a line revolving about a fixed point p_2, cut off on two other straight lines, revolving about two fixed points c_1 and c_2 collinear with p_2, constant distances $x\tau_1$ and $x\tau_2$ then x lies on a Cartesian oval.

For by the theorem of Menelaus

$$x\tau_2 \cdot \lambda_2 \cdot \rho_1 - x\tau_1 = \rho_2 - x\tau_2 \cdot x\tau_1 \cdot \lambda_1.$$

Or if we take the constants $x\tau_2$ and $x\tau_1$ to be $\sqrt{\lambda_1\lambda_3}$ and $\sqrt{\lambda_2\lambda_3}$ respectively,

$$\sqrt{\lambda_1\lambda_3} \cdot \lambda_2 \cdot \rho_1 - \sqrt{\lambda_2\lambda_3} = \rho_2 - \sqrt{\lambda_1\lambda_3} \cdot \sqrt{\lambda_2\lambda_3} \cdot \lambda_1.$$

Hence we have

$$\sqrt{\lambda_2}\rho_1 - \sqrt{\lambda_1}\rho_2 = \sqrt{\lambda_3}(\lambda_2 - \lambda_1).$$

Therefore x lies on the Cartesian oval.

Therefore we may define a Cartesian oval as the locus of the vertex of a triangle with two of its sides of fixed lengths and passing always through two fixed points while the base revolves about a fixed point collinear with the other two.

If we had measured the constant distances from f_1 and f_2 along the focal radii instead of from the point x we should have had Descartes' definition of the curve, namely, the locus of a point whose distances from the circumferences of two circles (with centers at f_1 and f_2) are in a constant ratio.

6. Intersections of Circle with the curve.

2/ $\qquad x = p(\alpha + v)$

and $\qquad \bar{x} = p(\alpha - v)$

are the parametric equations of the Cartesian oval, and the equation of any other curve is

$$R(x\,\bar{x}) = 0,$$

The intersections of the two curves are given by the zeros v_i of

$$f(v) = R\left[p(\alpha+v),\ p(\alpha-v)\right],$$

whose moduli are $\pm\alpha$.
Hence

(1) $\qquad \sum v_i \equiv 0$

from Abel's theorem for integrals of the first kind. [Picard, Tome II, p. 396.]

Let $z = \alpha + v$.

Then

$$\phi(z) - k = K[pz \cdot p(\alpha - z)] = A \prod \frac{\sigma(z - z_r)}{\sigma(z - c_r)},$$

where z_r is a zero of $\phi(z) - k$ and c_r is an infinity or a pole, of $\phi(z) - k$.

Accordingly

$$D \log K = \sum \zeta(z - z_r) - \sum \zeta(z - c_r).$$

But

$$D \log K = \frac{\phi'(z)}{\phi(z) - k}.$$

Hence, when $z = 0$

$$(2) \quad \frac{\phi'(0)}{\phi(0) - k} = -\sum \zeta z_r + \sum \zeta c_r,$$

which is a form of Abel's Theorem for elliptic integrals of the second class. [Hancock and Morley, Treatise on Theory of Func., p. 405 ff.] Since

$$\int u = -\int \wp u\, du = \int \frac{du\, d\wp u}{\sqrt{4\wp^3 u - g_2 \wp u - g_3}},$$

so that the curve is a circle with centre c. Its equation is

$$x\bar{x} - \bar{c}x - c\bar{x} + \lambda = 0,$$

and its intersections with our curve are given by the zeros of

$$\wp(x+v)\,\wp(x-v) - \bar{c}\,\wp(x+v) - c\,\wp(x-v) + \lambda = 0,$$

an elliptic function of the fourth order with double infinities at $\pm v$. Therefore

$$V_1 + V_2 + V_3 + V_4 = 0,$$

or the circle meets the curve in four points the sum of whose parameters

in 2 ways.

We proceed to apply equation (2) to find a second relation.

$$\phi(z) - k = pz \cdot p(2\alpha - z) - \bar{c}\, pz - c\, p(2\alpha - z) + \lambda - k$$

$$= A \prod \frac{\sigma(z - z_r)}{\sigma^2 z\, \sigma^2(2\alpha - z)}.$$

Therefore

$$\frac{\phi'}{\phi(z) - k} = \sum f(z - z_r) - 2fz - 2f(z - 2\alpha),$$

$$pz = \frac{1}{z^2} + 3S_4 z^2 - \cdots,$$

$$p(z - 2\alpha) = p2\alpha - z\, p'2\alpha + \frac{z^2}{2} p''2\alpha - \cdots,$$

$$\phi(z) - k = \frac{1}{z^2} p2\alpha - \frac{1}{2} p'2\alpha + \frac{z}{2} p''2\alpha - \cdots$$

$$- \bar{c}\left(\frac{1}{z^2} - \cdots\right) - c\, p2\alpha - \cdots + \lambda - k.$$

$$\frac{\sigma'z}{\varphi z - K} = \frac{-\frac{2}{z^3}(\wp 2\alpha - \bar{c}) + \frac{1}{z^2}\wp' 2\alpha + (P_0 z \cdots}{\frac{1}{z^2}(\wp 2\alpha - \bar{c}) - \frac{1}{z}\wp' 2\alpha + \frac{1}{2}\wp' 2\alpha - (\wp 2\alpha + \lambda - K)}$$

$$= \frac{-\frac{2}{z}(\wp 2\alpha - \bar{c}) + \wp' 2\alpha + z^2 P_0 z - \cdots}{\wp 2\alpha - \bar{c} - z\,\wp' 2\alpha - z^2(c\wp 2\alpha - \frac{1}{2}\wp' 2\alpha - \lambda + K)}$$

$$= \left[-\frac{2}{z} + \frac{\wp' 2\alpha}{\wp 2\alpha - \bar{c}} + \cdots\right]\left[1 + \frac{z\,\wp' 2\alpha}{\wp 2\alpha - \bar{c}} + \cdots\right]$$

Therefore

$$\sum f(z - z_r) - \frac{2}{z} \cdots - 2f(z - 2\alpha) = -\frac{2}{z} - \frac{\wp' 2\alpha}{\wp 2\alpha - \bar{c}} - (P z$$

Whence, if $z = 0$,

$$\sum_{1}^{H} f z_r - 2 f 2\alpha = \frac{\wp' 2\alpha}{\wp 2\alpha - \bar{c}}$$

or substituting for z its value $\alpha + \sigma$

$$\sum f(\alpha + \sigma) = 2 f 2\alpha + \frac{\wp' 2\alpha}{\wp 2\alpha - \bar{c}}.$$

Similarly,

$$\sum^{4} f(x-v) = 2f2x + \frac{f'2x}{f2x-C}.$$

These equations give the centre of the circle. If C is at infinity the circle is a line, and we have for the intersection of a line with a Quartic curve

$$\sum^{4} f(x+v) = 2f2x \qquad \text{and}$$
$$\sum^{4} f(x-v) = 2f2x.$$

Therefore

$$\sum^{4} f(\alpha+v_r) + \sum^{4} f(\alpha-v_r) - 4f2\alpha = 0,$$

or

$$\sum \left[f(\alpha+v_r) + f(\alpha-v_r) - f2\alpha \right] = 0.$$

But from page 36 we have

$$f(\alpha+v) + f(\alpha-v) - f2\alpha = \sqrt{2T},$$

where T is the distance from any point of the curve to the double tangent.

Therefore,

if a straight line be drawn cutting a Cartesian oval in four points the sum of the square roots of twice the distance of these points from the double tangent is zero.

But we found on page 87 that

$$\sqrt{2}\,T = \frac{\rho_1 - \sqrt{\lambda_2\lambda_3}}{\lambda_1}.$$

Hence, if ρ_{1r} denote the direct distance of any of the points $x+iz$ from the focus $\epsilon_1, 0$, we have

$$\sum \frac{\rho_{1r} - \sqrt{\lambda_2\lambda_3}}{\sqrt{\lambda_1}} = 0,$$

or

$$\sum \rho_{1r} = 4\sqrt{\lambda_2\lambda_3}.$$

Thereon the sum of the distances of the points in which a straight line cuts a Cartesian oval from any focus is constant, a well known property.

It follows as a corollary to this that the points of contact of the double tangent lie on a circle with centre e_1 and radius $\sqrt{k_2 k_3}$.

7 Bitangent Circles.

In the case of bitangent circles the equation

$$\Sigma v_i \equiv 0$$

reduces to

$$2v_1 + 2v_2 = 0$$

Four cases are to be distinguished,

when

$$v_2 = -v_1,$$

and when

$$v_2 = -v_1 + \omega_2$$

and the coincident point $v_2 = -v_1 - \omega_2 \quad (\lambda = 1, 2, 3)$

Hence there are four systems of bitangent circles.

In the first case,

$$\Sigma f(\alpha + v_1) = 2 f 2\alpha + \frac{p' 2\alpha}{p 2\alpha - \bar{c}}$$

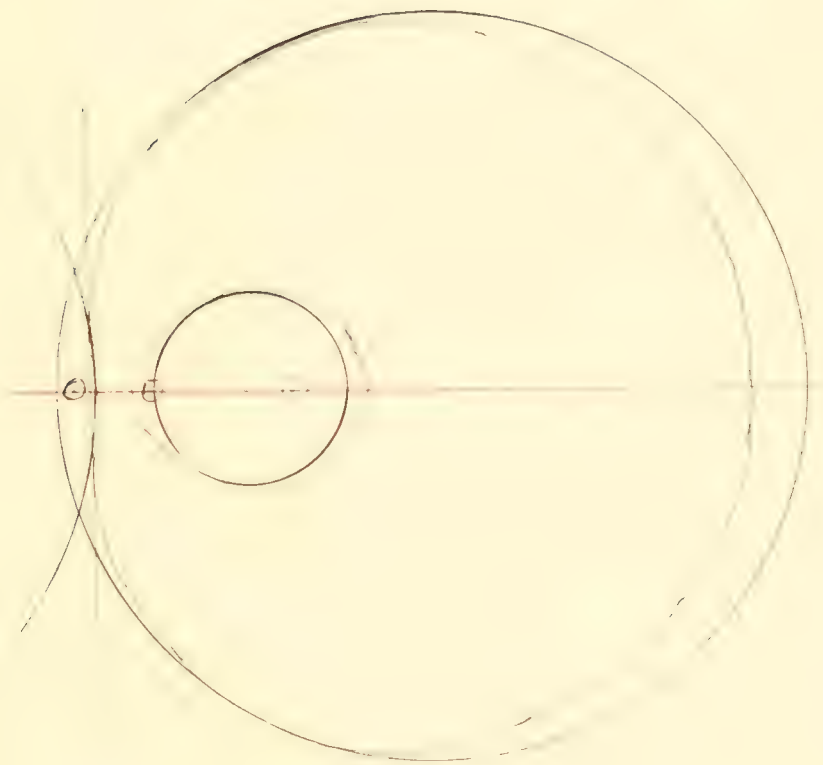
becomes

$$f(\alpha + v_1) + f(\alpha - v_1) - f 2\alpha = \frac{1}{2} \frac{p' 2\alpha}{p 2\alpha - \bar{c}},$$

And its conjugate

$$f(\alpha - v_1) + f(\alpha + v_1) - f 2\alpha = \frac{1}{2} \frac{p' 2\alpha}{p 2\alpha - c}.$$

Therefore $c = \bar{c}$, is the centric locus



on the real axis as is evident geometrically from the symmetry of the curve.

*Among the circles of this system are the bitangent line, two circles with four-fold contact with the inner oval, one at $\alpha + w$, and one at $\alpha + w_2$ and two with the outer oval, one at α_- and

*The relation of these bitangent circles is well brought out in a study of spheres-cones.

Suppose a sphere to be intersected by a quadric cone with vertex at the center. Through the intersection pass a pencil of quadrics among which are four degenerate quadrics or cones three bicircular

The other is $x+w_-$, and three point circles with imaginary contact, the three single foci. (The fourth point circle is at infinity

the cone already mentioned. These three cones have their vertices at infinity or are cylinders whose axes are mutually perpendicular. From one of the foci of the spherconic project the figure onto the plane of the great circle of which that focus is the pole.

The two ovals of the spherconic project into the two ovals of the Cartesian; the great circle on which the four foci lie into the real axis with the three foci lying on it and the fourth at infinity. Through each element of the four cones pass a tangent

In the other case we have

$$2f(\alpha+v)+f(\alpha-v+w_\lambda)+f(\alpha-v-w_\lambda)-2f_{2\alpha}=\frac{p'_{2\alpha}}{p_{2\alpha}-\bar{c}},$$

or

$$2f_{(\alpha+v)}+f_{(\alpha-v+w_\lambda)}+f_{(\alpha-v+w_\lambda-2w_1)}-2f_{2\alpha}=\frac{p'_{2\alpha}}{p_{2\alpha}-\bar{c}},$$

plane which intersects the sphere in a circle bitangent to one of the sphero-cones (or both) with real or imaginary contact. These project into bitangent lines of the Cartesian and we get the four systems corresponding to the four cones. The sphere and each of the other quadrics have four common tangent planes (imaginary in every case but one) and since these plane cut out point circles on the sphere, they touch it at the four real foci of the two sphero-cones which we have already said project into the four foci of the Cartesian and one being

or $f_{(\alpha+v)} + f_{(\alpha-\iota^{-}+\omega_\lambda)} - f_{\omega'_\lambda} - f_{2\alpha} = \frac{\frac{1}{2}\, p'_{2\alpha}}{p_{2\alpha} - C}$.

Likewise

$$f_{(\alpha-v)} + f_{(\alpha+\iota^{-}-\omega_\lambda)} + f_{\omega'_\lambda} - f_{2\alpha} = \frac{\frac{1}{2}\, p'_{2\alpha}}{p_{2\alpha} - C}.$$

---

at infinity), the circle cut out by the tangent plane through the focus, from which we are making the projection, to the plane which has real common tangent planes with the given, projects into a chord, at one end of the diameter the tangent line.

The tangent planes of the cone with vertex at the center, and also of the cylinder whose elements are parallel to the axis of the cone, determine circles whose projections touch different branches of the Cartesian curve, in one case on the same sides, in the other on opposite sides. [See Darboux. Sur une Classe Remarquable de Courbes et de Surfaces]

The following formula is one given by Professor Hensley. [Proc. London Math. Soc., S. 2, V. 4, p. 390]

$$f(u) + f(v) + f(w) - f(u+v+w) = \frac{\sigma(v+w)\,\sigma(u+w)\,\sigma(u+v)}{\sigma u \cdot \sigma v \cdot \sigma w \cdot \sigma(u+v+w)}$$

To prove it suppose $u + v + w$ a constant.

Then

$$f(u) + f(v) + f(w) - f(u+v+w)$$

is an elliptic function with two infinities,

$$u = 0$$

and $$u = -v - w$$

and two zeros,

$$u = -v$$

and $$u = -w.$$

Therefore it is equal to

$$K \frac{\sigma(u+v)\,\sigma(u+w)}{\sigma u\, \sigma(u+v+w)}.$$

To find K multiply both sides of the

equation by a and find the limit when $a = 0$.

We get

$$k = \frac{\sigma(b+c)}{\sigma b \cdot \sigma c}.$$

Therefore

$$\zeta u + \zeta v + \zeta c - \zeta(u+v+c) = \frac{\sigma(u+b)\sigma(b+c)\sigma(c+u)}{\sigma u \sigma v \sigma c \sigma(u+v+c)}.$$

Hence

$$\zeta(\alpha+v) + \zeta(\alpha-v+\omega_1) - \zeta\omega_1 - \zeta 2\alpha$$

$$= -\frac{\sigma(2\alpha+\omega_1)\,\sigma(\alpha+v-\omega_1)\,\sigma(\alpha-v)}{\sigma(\alpha+v)\,\sigma(\alpha-v+\omega_1)\,\sigma\omega_1\,\sigma 2\alpha} = \frac{\frac{1}{2}\,p'2\alpha}{p2\alpha - e},$$

and in the same way

$$\zeta(\alpha-v) + \zeta(\alpha+v-\omega_1) + \zeta\omega_1 - \zeta 2\alpha$$

$$= \frac{\sigma(2\alpha-\omega_1)\,\sigma(\alpha-v+\omega_1)\,\sigma(\alpha+v)}{\sigma(\alpha-v)\,\sigma(\alpha+v-\omega_1)\,\sigma\omega_1\,\sigma 2\alpha} = \frac{\frac{1}{2}\,p'2\alpha}{p2\alpha - e}.$$

Multiplying the two equations, we get

$$-\frac{\sigma(2\alpha+\omega_1)\,\sigma(2\alpha-\omega_1)}{\sigma^2\omega_1\,\sigma^2 2\alpha} = \frac{\frac{1}{4}\,p'^2 2\alpha}{p2\alpha - e \cdot p2\alpha - e}.$$

But the left hand member of this equation is equal to

$$p2\alpha - e_1 = \lambda_1.$$

[Hawkes and Morley, Introduction, §104.]

Therefore

$$\lambda_1 = \frac{\frac{1}{4}p'^2_{2\alpha}}{p2\alpha - \bar{c}\cdot p2\alpha - c} = \frac{\lambda_2\lambda_4\lambda_5}{p2\alpha - \bar{c}\cdot p2\alpha - c}.$$

Hence

$$(\bar{c} - p2\alpha)(c - p2x) = \lambda_4\lambda_5.$$

Therefore the locus of the centers of each of these three systems of bitangent circles is a circle whose center is $p2\alpha$ the triple focus, and whose radius is $\sqrt{\lambda_4\lambda_5}$.

For example, the locus of the centers of the bitangent circles touching the curve at $x+v$ and $x-v+w$, which circles with the triple focus as center and $\sqrt{\lambda_2\lambda_3}$ as radius.

That this set of circles is orthogonal to a circle with radius equal to

$$\sqrt{k_2 - L},\ \sqrt{k_3 - L},$$

and centre L, follows from the fact developed on page 17, that $p(x-l)+m$ and $p(x-l')+m$ are inverse points as to this circle, and the theorem of elementary geometry that the product of a secant to a circle from a point without and its external segment is equal to the square of the tangent from the same point.

Hence we get the theorem of Casey, in the Transactions of the Royal Irish Academy, (see [Williamson, Diff. Cal. p. 183]):

If a circle cut a given circle orthogonally, while its centre moves along another given circle its

envelope is a Cartesian oval".

Among the circles of the bitangent system orthogonal to the circle with ξ_1 as center is the circle touching the oval at α and $\alpha+\omega_1$, and the one touching it at $\alpha+\omega_2$ and $\alpha+\omega_3$. The center of the first circle is

$$\tfrac{1}{2}(\alpha+\alpha+\omega_1),$$

and of the second is

$$\tfrac{1}{2}(\alpha+\omega_2+\alpha+\omega_3).$$

Therefore the center of the circle of centers must be

$$\tfrac{1}{4}(\alpha+\alpha+\omega_1+\alpha+\omega_2+\alpha+\omega_3)$$

But we have just proved that this point is the triple focus.

Hence the triple focus is the Centroid of the four points in which the real axis cuts the Cartesians and which gives a simple method for the

Construction of the triple focus.

As a theorem for this construction it is found that, if a straight line cuts the curve in points whose parameters are v_r,

$$\Sigma f(\alpha + v_r) = 2 f 2\alpha ;$$

If the line is the line at ∞ we have

$$f\alpha + f(\alpha+\omega_1 + f(\alpha+\omega_2) + f(\alpha+\omega_3) = 2 f 2\alpha ;$$

Hence by differentiation

$$p\alpha + p(\alpha+\omega_1) + p(\alpha+\omega_2) + p(\alpha+\omega_3) = 4 p 2\alpha ,$$

or $p 2\alpha$ is the centroid of three points

whose Centres of the system Centres on the node with radius $\sqrt{\lambda_1 \lambda_2}$, orthogonal to the circle whose Centre is λ_3 and radius

$$\sqrt{\lambda_2 - \lambda_3} \cdot \sqrt{\lambda_1 - \lambda_3} ,$$

is the circle touching the curves $u-x$ and $x+w_3$ are one touching at $x-$, once $x+w_2$, also two twice point-point contact to the outer oval at the points

$$x \pm \tfrac{1}{2} w_3,$$

and two to the inner oval at

$$x + w_1 \pm \tfrac{1}{2} w_3 .$$

These are the points of contact of the tangents from e_3 to the curves; for since $p(x-v+w_3)$ is the inverse of $p(x+v$ as to the circle with centre e_1, mentioned above, if $v = \frac{1}{2} w_3$ we have

$$p(x - \tfrac{1}{2} w_3 + w_3) = p(x + \tfrac{1}{2} w_3)$$

the inverse of

$$p(x + \tfrac{1}{2} w_3)$$

that is, the inverse of itself.

Hence the points of contact of the tangent circle come together when the tangent circle with centre e_3 cuts

the curve and the focal vector-point e_3 is tangent at this point.

A similar proof holds for the pair

$$\alpha + w_1, \pm \frac{1}{e} e_3$$

The circle with e_3 as centre cuts both ovals orthogonally. Of course this, as the Cartesian oval corresponding to

$$\alpha = \tfrac{1}{2} w_1$$

in the 2-system

Hence it cuts every oval of the 2-system orthogonally.

The 1-system of tangent circles is the one with centre on the circle with centre $p\alpha$ and radius $\sqrt{e_1 e_3}$.

Among this system of circles are those touching the outer oval at α and the inner at $\alpha + w_2$ and another touching the outer at $\alpha + w_3$ and the inner at $\alpha + w_1$.

This figure shows the circles of contact (the red circles) and the two real circles of inversion (the black circles).

8. Relation of tangent to tangent circles

If $k\bar{x} - \bar{c}x - c\bar{x} + 1 = 0$

is the equation of a circle and

x_1 and $\bar{x}_1$

are the coordinates of a point and tangents are drawn from the point to the circle, the square of the distance from the point to the circle measured along the tangent is

$$x_1\bar{x}_1 - \bar{c}x_1 - c\bar{x}_1 + 1.$$

Hence if L be the distance from a point on the curve to a circle,

$$L = \sqrt{f(\alpha+\nu)\,f(\alpha-\nu) - \bar{c}\,p(\alpha+\nu) - c\,p(\alpha-\nu) + 1}$$

$$= \sqrt{A^2 \prod \frac{\sigma_1(\alpha+\cdots) - \sigma_1(\cdots)}{\sigma^2(\alpha+\sigma)\,\sigma^2(\alpha-\cdots)}}.$$

If the circle is a bitangent circle,

$$\mathfrak{L} = \sqrt{A^2 \frac{\sigma^2(\alpha+v+v_1)\,\sigma^2(\alpha+v-v_1)}{\sigma^2(\alpha+v)\,\sigma^2(\alpha-v)}}$$

$$= A\,\frac{\sigma(\alpha+v+v_1)\,\sigma(\alpha+v-v_1)}{\sigma(\alpha+v)\,\sigma(\alpha-v)}\,,$$

Similarly if $\mathfrak{L}'$ is the distance from the same point to a second bitangent circle of the same system,

$$\mathfrak{L}' = A'\,\frac{\sigma(\alpha+v+v_1')\,\sigma(\alpha+v-v_1')}{\sigma(\alpha+v)\,\sigma(\alpha-v)}\,.$$

Therefore $\mathfrak{L}$ and $\mathfrak{L}'$ are elliptic functions of the second order and the same poles, and if v_1 is the residue of $\mathfrak{L}$ at

$$v = \alpha$$

and v_1' is the residue of $\mathfrak{L}'$ at the same pole, then

$$r_1' x - r x' = h.$$

Tangent between tangent to one to tangent circle to the same system of a Cartesian oval there is a linear relation

$$r_1' x - r_1 x - h = 0.$$

Biographical Note.

Clara Latimer Bacon was born in 1866 at Hillsgrove, McDonough County, Illinois.

She was graduated from Normal University, Illinois in 1886.

After a year of teaching she entered Wellesley College, Wellesley, Massachusetts from which she received the A.B. degree in 1890.

She then taught one year in Kentucky and six years in Illinois in secondary schools and at Normal College.

From 1897–1905 she was instructor, and from 1905–1910 Associate Professor in Mathematics at Goucher College.

She studied Mathematics at the University of Chicago during the

summer quarter from 1901-1904 and in 1907 and 1908, taking the M. A. degree in 1904.

She entered the Johns Hopkins University as a student in Mathematics, Education and Philosophy, in October 1907. She was Fellow by courtesy in the University from 1909-1911, and for the year 1910-1911 has held a fellowship given by the Baltimore Association for the Promotion of the University Education of Women.

She is very glad to have this opportunity to express her gratitude to her instructors, especially to Professor Hayes of Wellesley, to the late Professor Macdonald and Professor Bolza of the University of Chicago, and to Professor Morley and Dr. Buchanan, Dean Griffin, Dr Cohen and Dr. Cook

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